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Finite nonsolvable groups whose character graphs have no triangles ☆

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ABSTRACT

In this paper, we show that the alternating group A_5 is the only nonsolvable group whose character graph has no triangles.

1. Introduction

There are several kind of graphs attached to finite groups, such as conjugacy graphs, degree graphs and character graphs. Various interesting results have been developed since they were introduced. For detailed information on these graphs we refer to a survey article [14] and other articles [4,13,15, 18–20,28].

In this paper, we shall focus on character graphs. Recall that the character graph $\Gamma(G)$ of a finite group G is defined in the following way. The vertices of the graph are the nonlinear complex irreducible characters of G, and there is an edge between two vertices χ and ψ if and only if $\chi(1)$ and $\psi(1)$ have a common prime divisor (see [19]).

In [19], the authors show that the number of connected components of the graph $\Gamma(G)$ is generally at most 3 and is at most 2 if G is solvable. Recently, in their paper [27] Yi-Tao Wu and Pu Zhang give a characterization of finite solvable groups whose character graphs have no triangles. Their result says that, for a finite nonabelian solvable group G, the character graph $\Gamma(G)$ has no triangles if

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and only if either the number of nonlinear irreducible characters of G is not greater than 2, or G is isomorphic to the symmetric group S_4 (see Theorem 1.1 in [27]). As mentioned in [27], the result fails for nonsolvable groups. For example, the alternating group A_5 has exactly four nonlinear irreducible characters of degrees $3, 3, 4, 5$, respectively. So the character graph of A_5 does not have a triangle. In this paper, we shall show that A_5 is the only exception for nonsolvable groups. The following theorem is our main result.

Theorem 1.1. *Let G be a finite nonsolvable group whose character graph has no triangles. Then $G \cong A_5$, the alternating group on five letters.*

According to the above theorem and Theorem 1.1 in [27], we can easily get that, for a finite group, its character graph has no triangles if and only if the graph doesn't contain a cycle, i.e. each connected component of the graph is a tree.

We notice that the analogous work has been done in [8] for conjugacy graphs. As for the definition of conjugacy graphs, see [1].

Now we sketch our proof of Theorem 1.1 as follows. First, using the classification theorem of finite simple groups, we show that A_5 is the only nonabelian simple group whose character graph contains no triangles (see Proposition 3.4). Then we prove that a group must be isomorphic to A_5 if it is perfect and its character graph has no triangles (see Proposition 4.3). In addition, we argue that a group must be perfect if it is nonsolvable and its character graph doesn't have a triangle (see Proposition 5.2). Finally, the main theorem follows from the above results.

At the end of the introduction, we introduce some notations. We mention that all groups considered in this paper are finite. Let G be a finite group. $Irr(G)$ is the set of its complex irreducible characters and $Irr_1(G)$ is the subset of $Irr(G)$ consisting of the nonlinear ones. $cd(G) = \{\chi(1) \mid \chi \in Irr(G)\}$ and $|cd(G)|$ denotes the number of its elements. Let $N \lhd G$ and let $\phi \in Irr(N)$. $Irr(\phi^G)$ denotes the set of irreducible constituents of ϕ^G in G. Suppose that n is an integer. If $n = p^k m$ where p is a prime and $(p, m) = 1$, n_p means the p-part p^k of n. If x is a positive real number, $[x]$ denotes the largest integer not greater than x. Finally, A_n denotes the alternating group on n letters. The notations used in this paper are basically standard (see [2,10]).

2. Preliminaries

In this section, we collect some facts needed to prove the main theorem. Exercise 6.3 and Corollary 11.29 in [10] imply the following facts respectively:

Lemma 2.1. *Let $N \lhd G$. Suppose that there is an irreducible character θ of N which is G-invariant. If $|G/N|$ is not a square of any integer, then θ^G has at least two irreducible constituents in G.*

Lemma 2.2. *Let G be a finite group and let N be a normal subgroup of G such that $p \nmid |G/N|$, where p is a prime. Then $\chi(1)_p = \theta(1)_p$ for any $\chi \in Irr(G)$ and $\theta \in Irr(N)$ such that $[\chi_N, \theta] \neq 0$.*

As a consequence, if G has k irreducible characters $\chi_1, \chi_2, \ldots, \chi_k$ such that $1 < \chi_1(1)_p < \chi_2(1)_p < \cdots < \chi_k(1)_p$, then N has k irreducible characters $\theta_1, \theta_2, \ldots, \theta_k$ such that $1 < \theta_1(1)_p < \theta_2(1)_p < \cdots < \theta_k(1)_p$. Moreover, $\theta_i(1)_p = \chi_i(1)_p$ for all $i = 1, 2, \ldots, k$.

The Ito–Michler Theorem [16] and the Feit–Thompson Theorem [7] imply the following result:

Lemma 2.3. *Let G be a finite group. Suppose that $\chi(1)$ is odd for each $\chi \in Irr(G)$. Then G has a normal abelian Sylow 2-subgroup and hence G is solvable.*

At the end of this section, we show that all the character graphs of the nonsolvable groups of order 120 have a triangle.

Lemma 2.4. *Let G be a nonsolvable group of order 120. Then the character graph of G has a triangle.*

Table 1

Lie type	Simple group classical	Atlas	Adjoint group	Index
A_ℓ	$PSL_{\ell+1}(q)$	$L_{\ell+1}(q)$	$PGL_{\ell+1}(q)$	$(\ell+1, q-1)$
${}^2A_\ell$	$PSU_{\ell+1}(q^2)$	$U_{\ell+1}(q)$	$PU_{\ell+1}(q)$	$(\ell+1, q+1)$
B_ℓ	$\Omega_{2\ell+1}(q)$	$O_{2\ell+1}(q)$	$SO_{2\ell+1}(q)$	$(2, q-1)$
C_ℓ	$PSp_{2\ell}(q)$	$S_{2\ell}(q)$	$PCSp_{2\ell}(q)$	2
D_ℓ	$P\Omega^+_{2\ell}(q)$	$O^+_{2\ell}(q)$	$P(CO_{2\ell}(q)^0)$	$(4, q^\ell-1)$
${}^2D_\ell$	$P\Omega^-_{2\ell}(q)$	$O^-_{2\ell}(q)$	$P(CO^-_{2\ell}(q)^0)$	$(4, q^\ell+1)$

Proof. All the nonsolvable groups of order 120 are the symmetric group S_5, $A_5 \times Z_2$, and the two cover $2.A_5$ of A_5. Since S_5 has nonlinear irreducible characters of degrees $4, 4, 5, 5, 6$, $A_5 \times Z_2$ has nonlinear irreducible characters of degrees $3, 3, 3, 3, 4, 4, 5, 5$, and $2.A_5$ has nonlinear irreducible characters of degrees $2, 2, 3, 3, 4, 4, 5, 6$, respectively, it follows that all their character graphs have a triangle. □

3. Character graphs of finite nonabelian simple groups

In this section, we check the nonabelian simple groups. As we will see, all the nonabelian simple groups except A_5 have a triangle.

Lemma 3.1. *For $n \geqslant 6$, the character graph of A_n has a triangle.*

Proof. By the character table of A_6 in the Atlas [3], it has nonlinear irreducible characters of degrees $5, 5, 8, 8, 9, 10$, respectively. So the character graph $\Gamma(A_6)$ has a triangle. For A_n ($n > 6$), it suffices to find three irreducible characters of A_n with even degrees.

If n is odd, then the characters corresponding to the non-self-associated partitions $(n-1, 1)$, $(2, 1^{n-2})$, and $(n-3, 3)$ are satisfied since the degrees of them are $n-1, n-1$, and $\frac{n(n-1)(n-5)}{3!}$, respectively. Here we reference [12] for the degree formula.

Suppose n is even. Then the characters corresponding to the non-self-associated partitions $(n-3, 2, 1)$ and $(3, 2, 1^{n-5})$ have the same even degree $\frac{n(n-2)(n-4)}{3}$. In order to find the third character, we consider the cases when $4|n$ and $4 \nmid n$, respectively. If $4|n$, then the character corresponding to the non-self-associated partition $(n-2, 2)$ has an even degree $\frac{n(n-3)}{2}$. If $4 \nmid n$, then the character corresponding to the non-self-associated partition $(n-3, 1^3)$ has an even degree $\frac{(n-1)(n-2)(n-3)}{3!}$. □

Lemma 3.2. *Let G be a finite simple group of Lie type. Suppose that $G \ncong PSL_2(4)$. Then the character graph of G has a triangle.*

Proof. Note that $PSL_2(4) \cong PSL_2(5) \cong A_5$. By checking the character tables in the Atlas [3], we have that all the character graphs $\Gamma(PSL_2(7))$, $\Gamma(PSL_3(2))$, $\Gamma(PSL_3(3))$, $\Gamma(PSL_3(4))$, $\Gamma(PSL_4(3))$, $\Gamma(PSU_4(2))$, $\Gamma(PSU_4(3))$, $\Gamma(PSU_5(2))$, and $\Gamma({}^2F_4(2)')$ have a triangle. Except these groups, it suffices to show that G has at least three irreducible characters whose degrees have a common prime divisor.

Case 1. *Classical groups of Lie type.* The classical groups of Lie type are the linear, unitary, symplectic, and orthogonal groups. Table 1 gives the notation of classical groups and the notation used in the Atlas [3] for these groups. Also, listed are the adjoint group and the index of the simple group in the adjoint group for each type. The notation for the adjoint groups is as in [2] (see [26, p. 7]).

Type A_ℓ: Linear groups. In this case, $G \cong PSL_{\ell+1}(q)$, where q is a prime power of p. In the following content of the proof, we will always use q to be a prime power of p.

According to the character table of $PSL_2(q)$ (see [5, Section 38]), for q odd and $q > 7$, $PSL_2(q)$ has at least one irreducible character of degree $q+1$ and has at least two irreducible characters of degree $q-1$. For q even and $q > 4$, $PSL_2(q)$ has at least three irreducible characters of degree $q-1$.

For $q > 4$, according to the character table of $PSL_3(q)$ in [22], $PSL_3(q)$ has irreducible characters degrees q^3, $q(q+1)$, $q(q^2+q+1)$.

For the rest, we investigate the corresponding adjoint groups.

Let $\ell \geqslant 3$. Then $PGL_{\ell+1}(q)$ has the character degrees

$$\chi^{(1,1,\ell-1)}(1) = q^3 \frac{(q^{l-1}-1)(q^l-1)}{(q-1)(q^2-1)}$$

$$\chi(1) = q \frac{(q^2-1)(q^3-1)\cdots(q^{l+1}-1)}{(q^2-1)(q^{l-1}-1)}$$

(see Lemma 2.2 in [24]).

Recall that the degree of the Steinberg character of G is equal to $|G|_p$ (see Section 6 in [2]). By Lemma 2.2, G has at least three irreducible characters whose degrees can be divided by p, including its Steinberg character.

Type ${}^2A_\ell$: *Unitary groups.* In this case, G is the projective special unitary group $PSU_{\ell+1}(q^2)$. Because $PSU_2(q^2) \cong PSL_2(q)$, we take $\ell \geqslant 2$. If $\ell = 2$, then we take $q > 2$, as $PSU_3(2^2)$ is not simple.

(1) The character table of $PSU_3(q^2)$ in [22] shows that $\{q^3, q(q-1), q(q^2-q+1)\} \subseteq cd(PSU_3(q^2))$. This fact implies that G has at least three irreducible characters whose degrees can be divided by p.

(2) For $\ell \geqslant 3$ and $(\ell, q) \neq (3,2), (3,3), (4,2)$, $PU_{l+1}(q^2)$ has the character degrees

$$\chi^{(1,1,\ell-1)}(1) = q^3 \frac{(q^{l-1}-(-1)^{l-1})(q^l-(-1)^l)}{(q+1)(q^2-1)}$$

$$\chi_1(1) = q \frac{(q^2-1)(q^3+1)\cdots(q^{l+1}-(-1)^{l+1})}{(q^2-1)(q^{l-1}-(-1)^{l-1})}$$

(see Lemma 2.3 in [24]).

By Lemma 2.2, G has at least three irreducible characters whose degrees can be divided by p, including its Steinberg character.

Type B_ℓ $(\ell \geqslant 2)$: *Odd dimensional orthogonal groups*. In this case, G is the simple odd dimensional orthogonal group $\Omega_{2\ell+1}(q)$.

If $\ell \geqslant 3$, then $SO_{2\ell+1}(q)$ has the character degrees

$$\chi^\alpha(1) = \frac{1}{2} q^4 \frac{(q^{l-2}-1)(q^{l-1}-1)(q^{l-1}+1)(q^l+1)}{(q^2-1)^2}$$

$$\chi_1(1) = q \frac{(q^4-1)(q^6-1)\cdots(q^{2(l-1)}-1)(q^{2l}-1)}{(q^{l-1}+1)}$$

(see [25, Lemma 2.2]).

By Lemma 2.2, G has at least three irreducible characters whose degrees can be divided by p, including its Steinberg character.

Let $\ell = 2$. Then $G \cong B_2(q) \cong C_2(q)$, hence $G \cong PSp_4(q)$. Note that $B_2(2)$ is isomorphic to the symplectic group $Sp_4(2)$, which is not simple, we assume $q > 2$. If $q = 3$, then, by the character table in [21], G has more than three irreducible characters whose degrees can be divided by 3.

As for $q > 3$, the character tables in [21] and [6] or the proof of Theorem 5.7 in [26] shows that G has the character degrees $\chi_1(1) = q(q-1)(q^2+1)$ and $\chi_2(1) = q(q+1)(q^2+1)$. Therefore, combined with its Steinberg character, G has at least three irreducible characters whose degrees can be divided by p.

Type C_ℓ $(\ell \geqslant 3)$: *Symplectic groups.* In this case, G is the simple symplectic group $PSp_{2\ell}(q)$. Since $C_\ell(q) \cong B_\ell(q)$ for all ℓ if q is even, we may assume q is odd.

If $\ell \geqslant 4$, then the projective conformal symplectic group $PCSp_{2\ell}(q)$ has character degrees

$$\chi^{\alpha}(1) = q^3 \frac{(q^{2(l-2)}-1)(q^{2l}-1)}{(q^2-1)^2}$$
$$\chi_1(1) = q^2 \frac{(q^2+1)(q^6-1)(q^8-1)\cdots(q^{2(l-1)-1})(q^{2l}-1)}{q^{l-2}+1}$$

(see Lemma 2.3 in [25]).

By Lemma 2.2, G has at least three irreducible characters whose degrees can be divided by p, including its Steinberg character.

Suppose $\ell = 3$. Then $G \cong PSp_6(q)$. Since q is odd, the character table of G is available in the CHEVIE system (see [9]). The character table in [9] shows that G has more than three irreducible characters whose degrees can be divided by p.

Type D_ℓ and ${}^2D_\ell$ ($\ell \geqslant 4$): Even dimensional orthogonal groups. In this case, G is either $P\Omega^+_{2\ell}(q)$ or $P\Omega^-_{2\ell}(q)$. We first consider the case that $G \cong P\Omega^+_{2\ell}(q)$.

If $\ell \geqslant 6$, then $P((CO_{2\ell}(q))^0)$ has the character degrees

$$\chi^{\alpha}(1) = q^6 \frac{(q^{l-4}+1)(q^{2(l-3)}-1)(q^{2(l-1)}-1)(q^l-1)}{(q^2-1)^2(q^4-1)}$$
$$\chi_1(1) = q^2 \frac{(q^2+1)(q^6-1)\cdots(q^{2(l-2)-1})(q^{2(l-1)}-1)(q^l-1)}{(q+1)(q^{l-3}+1)}$$

(see Lemma 2.4 in [25]).

By Lemma 2.2, G has at least three irreducible characters whose degrees can be divided by p, including its Steinberg character.

Let $\ell = 5$. Then $P(CO_{2\ell}(q)^0)$ has the character degrees

$$\chi^{\alpha} = \frac{q(q^3-1)(q^2+1)}{q^2-1}$$
$$\chi^{\beta} = q^2\big(q^3-1\big)\big(q^2+1\big)$$

where α, β are the symbols $\binom{2}{1}$ and $\binom{1\ 2}{0\ 1}$.

Finally, let $\ell = 4$. If $q = 2$, then G has more than 3 irreducible characters whose degrees are even by the character table of G in the Atlas [3]. If $q = 3$, then again by the character table in the Atlas [3] G has more than 3 irreducible characters whose degrees are divided by 3. If $\ell = 4$ and $q > 3$, then $P((CO_8(q))^0)$ has the character degrees

$$\chi_1(1) = q^2(q-1)^2\big(q^2+q+1\big)\big(q^2+1\big)^2\big(q^2-q+1\big)$$
$$\chi^{\beta}(1) = \frac{1}{2}q^3(q+1)^4\big(q^2-q+1\big)$$

(see [25, Lemma 2.4]).

By Lemma 2.2, G has at least three irreducible characters whose degrees can be divided by p, including its Steinberg character.

Now we consider the case when $G \cong P\Omega^-_{2\ell}(q)$. If $\ell \geqslant 5$, then $P(CO^-_{2\ell}(q))^0$ has the character degrees

$$\chi^{\alpha}(1) = \frac{1}{2}q^3 \frac{(q^{l-3})(q^{l-2}+1)(q^{l-1}-1)(q^l+1)}{(q^2+1)(q-1)^2}$$

$$\chi_1(1) = q^2 \frac{(q^2+1)(q^6-1)\cdots(q^{2(l-2)}-1)(q^{2(l-1)}-1)(q^l+1)}{q^{l-2}+1}$$

(see Lemma 2.5 in [25]).

If $\ell = 4$, then $P(CO_8^-(q))^0$ has the character degrees

$$\chi_1(1) = q^2(q-1)(q+1)\left(q^2+q+1\right)\left(q^2-q+1\right)\left(q^4+1\right)$$
$$\chi_2(1) = q^3(q-1)\left(q^2+q+1\right)\left(q^2+1\right)\left(q^4+1\right)$$

(see Lemma 2.5 in [25]).

Anyway, by Lemma 2.2, G has at least three irreducible characters whose degrees can be divided by p, including its Steinberg character.

Case 2. *Exceptional groups of Lie type.* Let G be one of the groups ${}^3D_4(q^3)$, $G_2(q)$, $F_4(q)$, ${}^2B_2(q^2)$, $E_6(q)$, ${}^2E_6(q^2)$, $E_7(q)$, $E_8(q)$, ${}^2G_2(q^2)$, and ${}^2F_4(q^2)$. According to the unipotent characters of Section 13.9 in [2], G has at least three irreducible characters whose degrees can be divided by p. $\square$

Lemma 3.3. *Let G be one of the sporadic groups. Then the character graph $\Gamma(G)$ of G has a triangle.*

Proof. This is obvious by checking the character tables in the Atlas [3]. $\square$

According to the classification theorem of finite simple groups, the above three lemmas lead to the following result:

Proposition 3.4. *Let G be a finite nonabelian simple group. Suppose that $G \ncong A_5$. Then the character graph of G has a triangle.*

4. Perfect groups whose graphs contain no triangles

In this section, we prove that a group must be isomorphic to A_5 if it is perfect and its character graph doesn't have a triangle. The proof is based on the following lemmas.

Lemma 4.1. *Let G be a finite group with a chief series: $G \rhd G_1 \rhd 1$, where $G/G_1 \cong A_5$ and G_1 is nonabelian. Then the character graph of G has a triangle.*

Proof. Note that $Irr(G/G_1) \subseteq Irr(G)$ and that $Irr(A_5) = \{1, \varphi_1, \varphi_2, \varphi_3, \varphi_4\}$, where $\varphi_1(1) = \varphi_2(1) = 3$, $\varphi_3(1) = 4$, and $\varphi_4(1) = 5$.

We first consider the case when G_1 is a nonabelian simple group. By the classification theorem of finite simple groups, G_1 is isomorphic to one of the following groups: A_n ($n \geqslant 5$), simple groups of Lie type, or sporadic groups.

Case 1. $G_1 \cong A_n$ for some integer n ($n \geqslant 5$). By Lemma 6.8 in [10], $\theta(1) | \chi(1)$ for any $\theta \in Irr(G_1)$ and $\chi \in Irr(G)$ such that $[\chi_{G_1}, \theta] \neq 0$. Therefore, except φ_1 and φ_2 it suffices to find an irreducible character of G_1 whose degree can be divided by 3. But this is obvious by the following facts:

(1) $n \equiv 0 \bmod 3$. The character corresponding to the non-self-associated partition $\lambda = (n-2, 2)$ has degree $\frac{n(n-3)}{2}$ which can be divided by 3.
(2) $n \equiv 1 \bmod 3$. The character corresponding to the non-self-associated partition $\lambda = (n-1, 1)$ has degree $n-1$ which can be divided by 3.
(3) $n \equiv 2 \bmod 3$. The character corresponding to the non-self-associated partition $\lambda = (n-2, 1, 1)$ has degree $\frac{(n-1)(n-2)}{2}$ which can be divided by 3.

Case 2. G_1 is isomorphic to one of the simple groups of Lie type of characteristic p. In this case, consider the Steinberg character St_{G_1} of G_1. Noting that St_{G_1} is the unique irreducible character of G_1 such that $St_{G_1}(1) = |G_1|_p$, we have that St_{G_1} is G-invariant. Since $|G/G_1| = 60$ is not a square of any integer, according to Lemma 2.1 $St_{G_1}^G$ has at least two irreducible constituents whose degrees can be divided by p.

By the proof of Lemma 3.2, we may investigate G_1 case by case in the following way:

(1) If $G_1 \ncong PSL_2(q)$, then except its Steinberg character G_1 has another irreducible character whose degree can be divided by p. This implies that G has at least three irreducible characters whose degrees can be divided by p.
(2) If $G_1 \cong PSL_2(2^n)$, then $2|St_{G_1}(1)$ and $(St_{G_1})^G$ has at least two irreducible constituents of even degrees. Combined with φ_3, this implies that G has at least three irreducible constituents of even degrees.
(3) Let $G_1 \cong PSL_2(q)$ (q is odd and $q > 3$). In this case, G_1 has three irreducible characters $\theta_1, \theta_2, \theta_3$ satisfying $\theta_1(1) = q - 1$, $\theta_2(1) = q + 1$, and $\theta_3(1) = \frac{1}{2}(q + (-1)^{\frac{q-1}{2}})$. Without loss of generality we may assume that $(-1)^{\frac{q-1}{2}} = -1$. If one of θ_1 and θ_3 is G-invariant, according to Lemma 2.1 we deduce that G has at least three irreducible characters whose degrees can be divided by $\frac{p-1}{2}$. So we may assume that θ_1 is not G-invariant. If $I_G(\theta_1) \neq G_1$, then $\theta_1^G \notin Irr(G)$ and θ_1^G has at least two irreducible constituents by Theorem 6.11 and Problems 6.2, 6.3 in [10]. Hence G has at least three irreducible characters whose degrees can be divided by $\frac{p-1}{2}$. If $I_G(\theta_1) = G_1$, then $\theta_1^G \in Irr(G)$ and G has three irreducible characters φ_1, φ_2 and θ_1^G of even degrees.

Case 3. G_1 is isomorphic to one of the sporadic simple groups. It suffices to find an irreducible character of G_1 whose degree is greater than 3 and is divided by 3. This is true by checking the character tables in the Atlas [3]. Thus the lemma holds when G_1 is a nonabelian simple group.

Now we consider the case when G_1 is a direct product of some isomorphic nonabelian simple groups. Set $G_1 \cong S \times S \times \cdots \times S$, where S is a nonabelian simple group. Then $Irr(G_1) = Irr(S) \times Irr(S) \times \cdots \times Irr(S)$. Using the same method as above, we conclude that the lemma holds. □

Lemma 4.2. *Let G be a finite group with a chief series: $G \rhd G_1 \rhd 1$, where $G/G_1 \cong A_5$ and G_1 is an elementary abelian p-group. Then the character graph of G has a triangle.*

Proof. Note that $Irr(G/G_1) \subseteq Irr(G)$ and that $Irr(A_5) = \{1, \varphi_1, \varphi_2, \varphi_3, \varphi_4\}$, where $\varphi_1(1) = \varphi_2(1) = 3$, $\varphi_3(1) = 4$, and $\varphi_4(1) = 5$. Set $|G_1| = p^n$. The set $Irr(G_1)$ may be partitioned into the following two subsets:

$X_1 = \{\lambda_1 = 1_{G_1}, \lambda_2, \ldots, \lambda_r\}$, where λ_i is G-invariant for $i = 1, 2, \ldots, r$.

$X_2 = \{\xi_1, \xi_2, \ldots, \xi_s\}$, where ξ_j is not G-invariant for $j = 1, 2, \ldots, s$.

Set $Irr(X_1'^G) = \bigcup_{i=2}^r Irr(\lambda_i^G)$ and $Irr(X_2^G) = \bigcup_{j=1}^r Irr(\xi_j^G)$.

If there is a $\chi \in Irr(X_1'^G) \cup Irr(X_2^G)$ such that $3|\chi(1)$, then the lemma holds. So we may assume that $3 \nmid \chi(1)$ for any $\chi \in Irr(X_1'^G) \cup Irr(X_2^G)$. Then all such χ are nonlinear by Corollary 6.17 in [10]. Note that $\chi(1)||A_5| = 60$. Hence either $2|\chi(1)$ or $5|\chi(1)$.

We claim that there are at least two irreducible characters $\chi_1, \chi_2 \in Irr(X_1'^G) \cup Irr(X_2^G)$ such that $p|\chi_1(1)$ and $p|\chi_2(1)$, where $p = 2$, or 5. From this, we can easily deduce that the character graph of G has a triangle.

Case 1. $|X_1| > 2$. In this case, $|Irr(\lambda_2^G)| \geqslant 2$ and $|Irr(\lambda_3^G)| \geqslant 2$ by Lemma 2.1. So the claim holds.

Case 2. $|X_1| = 2$. If X_2 is empty, then $|G| = 120$ and the character graph of G has a triangle by Lemma 2.4. If X_2 is not empty, then $|Irr(\lambda_2^G)| \geqslant 2$ by Lemma 2.1. Notice that $|Irr(\xi_1^G)| \geqslant 1$. We deduce that the claim also holds in this case.

Case 3. $|X_1| = 1$. Assume that the claim doesn't hold in this case. Then $|Irr(X_1'^G)| + |Irr(X_2^G)| \leqslant 2$ and either $\chi(1)|4$ or $\chi(1) = 5$ for any $\chi \in Irr(X_1'^G) \cup Irr(X_2^G)$. Hence $60 \times p^n = |G| = \sum_{\chi \in Irr(G)} \chi(1)^2 \leqslant |G/G_1| + 4^2 + 5^2$. But it follows that $60 \times (p^n - 1) < 41$, a contradiction. Thus the claim holds. $\square$

Proposition 4.3. *Let G be a perfect group whose character graph contains no triangles. Then $G \cong A_5$.*

Proof. If G is simple, then the theorem holds by Proposition 3.4. Supposing that G is not simple, we deduce a contradiction.

Let $G \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_k = 1$ be a chief series of G. Noting that $\Gamma(G/G_i)$ may be viewed as a subgraph of $\Gamma(G)$ for $i = 1, 2, \ldots, k-1$, we may assume that $G_2 = 1$. Then G/G_1 is simple and is nonabelian by the assumption and the graph of G/G_1 also contains no triangles. By Proposition 3.4, $G/G_1 \cong A_5$. But it then follows that the character graph $\Gamma(G)$ of G has a triangle by Lemmas 4.1 and 4.2, which contradicts the assumption. $\square$

5. Nonsolvable groups whose graphs have no triangles

In this section, we argue that a group must be perfect if it is nonsolvable and its character graph doesn't have a triangle. In order to prove it we first show a lemma.

Lemma 5.1. *Let G be a finite nonsolvable group satisfying $|cd(G)| = 4$. Suppose that G is not isomorphic to A_5. Then either $|Irr_1(G)| > 5$ or the character graph of G contains the character graph of S_5 as a subgraph.*

Proof. By Theorem *A* in [17] and the hypothesis that $G \ncong A_5$, one of the following cases holds:

(i) $G \cong PSL_2(4) \times A$, where A is a nontrivial abelian group.
(ii) There is a normal subgroup N such that G/N is isomorphic to M_{10}, $PSL_2(2^n)$ for some $n > 2$, or $PGL_2(q)$ for some odd q ($\geqslant 5$), where M_{10} is the stabilizer of a point in the Mathieu group M_{11} in its natural permutation representation.

Suppose that $G/N \ncong PGL_2(5)$ ($\cong S_5$). We shall show that $|Irr_1(G)| > 5$.

As for (i), it is obvious that $|Irr_1(G)| > 5$. Consider the case (ii). Since $Irr(G/N) \subseteq Irr(G)$, it suffices to show that $|Irr_1(M_{10})|$, $|Irr_1(PSL_2(2^n))|$, and $|Irr_1(PGL_2(q))|$ are all greater than 5 for $n > 2$ and for odd $q(\geqslant 5)$. We investigate them case by case.

(1) M_{10}. Note that M_{10} has a normal subgroup M which is isomorphic to A_6. According to the Atlas [3], $M_{10} \cong A_6 \cdot 2_3$ and $|Irr_1(M_{10})| = 6$. Indeed, M_{10} has exactly six nonlinear irreducible characters of degrees 9, 9, 10, 10, 10, 16, respectively.

(2) $PSL_2(2^n)$ for some $n > 2$. By checking the character table of $PSL_2(2^n)$ (see [5, Section 38]), we have that $|Irr_1(PSL_2(2^n))| = 2^n > 5$.

(3) $PGL_2(q)$ for some odd q (> 5). By the character table III in [23], $PGL_2(q)$ has at least $2 + [\frac{1}{2}(q-3)] + [\frac{1}{2}(q-1)] = q$ nonlinear irreducible characters. So $PGL_2(q)$ has at least 6 nonlinear irreducible characters for $q > 5$. $\square$

Proposition 5.2. *Let G be a nonsolvable group whose character graph contains no triangles. Then G is perfect.*

Proof. Suppose that G is a minimal counterexample to the proposition. Then G has a normal subgroup M such that $|G/M| = p$ for some prime p. According to the Clifford theory for characters [10], the set $Irr(M)$ may be partitioned into the following four subsets:

$Y_1 = \{\lambda_1 = 1_M, \lambda_2, \ldots, \lambda_r\}$, where $\lambda_i(1) = 1$ and λ_i is G-invariant for $i = 1, 2, \ldots, r$.

$Y_2 = \{\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1p}, \lambda_{21}, \lambda_{22}, \ldots, \lambda_{2p}, \ldots, \lambda_{s1}, \lambda_{s2}, \ldots, \lambda_{sp},\}$, where $\lambda_{ij}(1) = 1$ and $\lambda_{i_1 j_1}$ and $\lambda_{i_2 j_2}$ are G-conjugate if and only if $i_1 = i_2$.

$Y_3 = \{\varphi_1, \varphi_2, \ldots, \varphi_t\}$, where $\varphi_i(1) \neq 1$ and φ_i is G-invariant for $i = 1, 2, \ldots, t$.

$Y_4 = \{\theta_{11}, \theta_{12}, \ldots, \theta_{1p}, \theta_{21}, \theta_{22}, \ldots, \theta_{2p}, \ldots, \theta_{u1}, \theta_{u2}, \ldots, \theta_{up}\}$, where $\theta_{ij}(1) \neq 1$ for $i = 1, 2, \ldots, u$; $j = 1, 2, \ldots, p$. Moreover, $\theta_{i_1 j_1}$ and $\theta_{i_2 j_2}$ are G-conjugate if and only if $i_1 = i_2$.

By Theorem 6.17 and Corollaries 6.19, 6.20 in [10], $Irr(\phi^G) = \{\alpha\beta \mid \alpha \in Irr(G)$ such that $[\alpha_M, \phi] \neq 0$, $\beta \in Irr(G/M)\}$ for $\phi \in Y_1 \cup Y_3$ and $\psi^G \in Irr(G)$ for $\psi \in Y_2 \cup Y_4$.

Since G is nonsolvable, it follows that M is also nonsolvable. Hence $Y_3 \cup Y_4$ is not empty.

Step 1. We claim that Y_3 is not empty. Otherwise, Y_4 must be not empty. Noting that $|cd(M)| > 3$ by Theorem 12.15 in [10], we have $u \geqslant 3$. Then it follows that G has at least three irreducible characters whose degrees are divisible by p. This implies that G has a triangle, which contradicts the hypothesis.

Step 2. We claim that $p = 2$. Note that Y_3 is not empty. If p is an odd prime, then G has at least p irreducible characters which have equal degrees. Therefore, it follows that G has a triangle, a contradiction.

As a consequence, $|G/G'| = 2^m$ for some integer m.

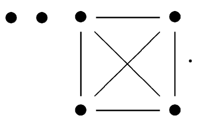
Step 3. We claim that m must be 1 and hence $G' = M$. Assume that $m \geqslant 2$. By the hypothesis $Irr_1(G)$ has at most two irreducible characters of even degrees. If G doesn't have nonlinear irreducible characters of odd degrees, we have $|cd(G)| \leqslant 3$. By Theorem 12.15 in [10], G is solvable, a contradiction. So G has at least one nonlinear irreducible character of odd degree. Set $\eta \in Irr(G)$ such that $\eta(1)$ is odd and $\eta(1) > 1$. Then $\eta_{G'} \in Irr(G')$ by Corollary 11.29 in [10]. Thus G has at least four irreducible characters of degree $\eta(1)$ by Corollary 6.17 in [10], which implies the character graph has a triangle, a contradiction.

Step 4. We claim that Y_4 is not empty. Suppose that Y_4 is empty. Then, by Lemma 2.3, there is an element of Y_3 of an even degree since M is nonsolvable. If Y_2 is not empty, then G has at least three irreducible characters whose degrees are divisible by 2. This implies that G has a triangle, a contradiction. If Y_2 is empty, then all the restrictions of irreducible characters of G to M are irreducible by Corollary 6.19 in [10]. Thus the character graph of M contains no triangles. Considering the minimality of G, M must be perfect and then $M \cong A_5$ by Proposition 3.4. Hence $|G| = 120$ and G has a triangle by Lemma 2.4, which contradicts the assumption.

Step 5. Final contradiction.

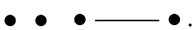
Let $n(\Gamma(M))$ be the number of the connected components of the character graph $\Gamma(M)$. Then $n(\Gamma(M)) \leqslant 3$ (see [19]). Noting that $|cd(M)| \geqslant 4$, we deduce that the following cases are all the possible ones.

(i) $u \geqslant 3$.
(ii) $u = 1, 2$ and $2|\varphi_i(1)$ for some $i = 1, 2, \ldots, t$.
(iii) $t > 1$ and $(\varphi_i(1), \varphi_j(1)) \neq 1$ for some pair (i, j), where $i, j = 1, 2, \ldots, t$ but $i \neq j$.
(iv) The character graph of M is of the following form:



Moreover, $Irr(M) = \{1, \varphi_1, \varphi_2, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}\}$, where $\varphi_1(1)$, $\varphi_2(1)$ are odd and coprime, $\theta_{11}(1) = \theta_{12}(1)$ is even, and $\varphi_1(1)$, $\varphi_2(1)$ are coprime to $\theta_{11}(1)$ and $\theta_{21}(1)$.

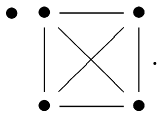
(v) The character graph of M is of one of the following types:

(a) 

Moreover, $Irr(M) = \{1, \varphi_1, \varphi_2, \theta_{11}, \theta_{12}\}$ where $\varphi_1(1), \varphi_2(1)$ are odd and coprime and $\theta_{11}(1) = \theta_{12}(1)$ is even.

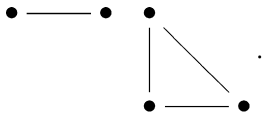
(b) 

Moreover, $Irr(M) = \{1, \varphi_1, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}\}$, where $\varphi_1(1)$ is odd, $\theta_{11}(1) = \theta_{12}(1)$ and $\theta_{21}(1) = \theta_{22}(1)$, and $\varphi_1(1)$, $\theta_{11}(1)$, and $\theta_{21}(1)$ are coprime to each other.

(c) 

Moreover, $Irr(M) = \{1, \varphi_1, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}\}$, where $\varphi_1(1)$ is odd, $\theta_{11}(1) = \theta_{12}(1)$ and $\theta_{21}(1) = \theta_{22}(1)$, and $\varphi_1(1)$ is coprime to $\theta_{11}(1)$ and $\theta_{21}(1)$.

Note that $M \ncong A_5$ as shown above. Notice that the character graph of S_5 is



Lemma 5.1 shows that (v) cannot happen. As for the former three cases, it is easy to see that each one of (i), (ii), and (iii) implies that the character graph of G has a triangle, a contradiction. So it remains to show that (iv) cannot happen. Assume that M satisfies (iv). We consider the cases when M is perfect and M is not perfect, respectively.

Case 1. $M \neq M'$. Set $\bar{G} = G/M'$. Noting that $G' = M$, we have $\bar{G}$ is not abelian. Since $\overline{M}$ is abelian, it follows that $\bar{G}$ has an irreducible character γ of degree $|\bar{G}/\overline{M}| = |G/M| = 2$ by a result of Ito (see [11]). Then the subgraph consisting of $\gamma, \theta_{11}^G, \theta_{21}^G$ is a triangle, which contradicts the assumption.

Case 2. M is perfect. Then M has a normal subgroup L such that M/L is nonabelian simple. Since $Irr(M/L) \subseteq Irr(M)$, it follows that $|cd(M/L)| = 4$ or 5. By Theorem C in [17], $M/L \cong PSL_2(2^n)$ $(n \geqslant 2)$ or $PSL_2(p^n)$ for p odd and $p^n > 5$. By the character table of $PSL_2(p^n)$ in Section 38 in [5], $|Irr_1(PSL_2(2^n))| = 1 + \frac{1}{2}(2^n - 2) + 2^{n-1} > 6$ for $n > 2$ and $|Irr_1(PSL_2(p^n))| = 1 + [\frac{1}{4}(p^n - 3)] + [\frac{1}{4}(p^n - 1)] + 2 > 6$ for p odd and $p^n \geqslant 11$. So $M/L \cong PSL_2(4), PSL_2(7)$ or $PSL_2(9)$.

According to the character tables in [10, p. 289], $PSL_2(9)$ has exactly six nonlinear irreducible characters of degrees $5, 5, 9, 10, 8, 8$, respectively. Hence the character graph of $PSL_2(9)$ is not a subgraph of M. Therefore, we have $M/L \ncong PSL_2(9)$.

Again by the character tables in [10, p. 289], $PSL_2(7)$ has exactly five nonlinear irreducible characters of degrees $6, 7, 8, 3, 3$, respectively. According to the statement (iv), it follows that M has exactly six nonlinear irreducible characters of degrees $k, 6, 7, 8, 3, 3$, respectively, where k is an odd integer and is coprime to 7. But this also contradicts the statement (iv) because $\theta_{11}(1) = \theta_{12}(1)$ is even in (iv).

Finally, suppose that $M/L \cong PSL_2(4) \cong A_5$. Note that A_5 has exactly four nonlinear irreducible characters of degrees $3, 3, 4, 5$, respectively. It follows that $\theta_{11}(1) = 4$ and $\theta_{21} = 3$, which implies the character graph doesn't have a triangle, a contradiction. Thus the proof is completed. □

6. The proof of the main theorem

Proof. This is obvious by Propositions 3.4, 4.3, and 5.2. □

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